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Enter AG

Samplingtheorem
Tutorial

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SAMPLINGTHEOREM

Tutorial

1 MOTIVATION

In theory we often encounter a continuous Fourier transform or a continuous convolution. In practice we should like to compute it as a finite, discrete Fourier transform or as a finite, discrete convolution. I have prepared this small tutorial more than thirty years ago to help me visualize the relationship between the continuous and the discrete versions. Thus it has the character of a memo to myself.

2 GLOBAL DEFINITIONS

In this tutorial we assume that all finite, discrete Fourier transforms and convolutions have fixed length $n \in \mathbb{N}$.

We use i for the imaginary constant $i=\sqrt{-1}$. Therefore we avoid using it as an index in the discrete cases.

Finally we use the abbreviation ω for the n -th root of unity:

$$\omega = e^{\frac{2\pi i}{n}} .$$

3 THE CONTINUOUS FOURIER TRANSFORM

We start out with two continuous complex functions f and g , which are a Fourier transform pair: $f, g \mathbb{R} \rightarrow \mathbb{C}$.

$$g(\nu) = \int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i \nu t} dt ,$$

$$f(t) = \int_{-\infty}^{\infty} g(\nu) \cdot e^{2\pi i \nu t} d\nu .$$

4 PERIODIC CONTINUATION

For the transition from the continuous Fourier transform to the (infinite) discrete Fourier sequences we need periodic functions. We choose periods T and N such that $T \cdot N = n$.

We define the periodic continuations of f and g as

$$f_p(t) = \sum_{m=-\infty}^{\infty} f(t + m \cdot T) \text{ and}$$

$$g_p(\nu) = \sum_{m=-\infty}^{\infty} g(\nu + m \cdot N) .$$

(For this not to diverge f and g must be square integrable or some such.)

5 SAMPLING OF PERIODIC FUNCTIONS

With f_p having period T and g_p having period N and $T \cdot N = n$ we define the time sample interval $\Delta t = \frac{1}{N}$ and the frequency sample interval $\Delta \nu = \frac{1}{T}$.

We further define the sequences $\{x_j\}_{j \in I}$, $\{\xi_j\}_{j \in \mathbb{Z}}$, $\{y_k\}_{k \in I}$ and $\{y_k\}_{k \in \mathbb{Z}}$ of samples with $I = [-\frac{n}{2}, \frac{n}{2}) \in \mathbb{Z}$ by

$x_j = f_p(j \cdot \Delta t)$ with $j \in I$,

$\xi_j = f(j \cdot \Delta t)$ with $j \in \mathbb{Z}$,

$y_k = g_p(k \cdot \Delta \nu)$ with $k \in I$ and

$\eta_k = g(k \cdot \Delta \nu)$ with $k \in \mathbb{Z}$.

With $j \cdot \Delta t = \frac{j}{N}$ we have

$$\xi_j \;=\; f(j \cdot \Delta t) \;=\; \int_{-\infty}^{\infty} g(\nu) \cdot e^{2\pi i \frac{j\nu}{N}} d\nu \;=\; \sum_{m=-\infty}^{\infty} \int_{(m-\frac{1}{2}) \cdot N}^{(m+\frac{1}{2}) \cdot N} g(\nu) \cdot e^{2\pi i \frac{j\nu}{N}} d\nu \;=\; \int_{\frac{-N}{2}}^{\frac{N}{2}} g_p(\nu) \cdot e^{2\pi i \frac{j\nu}{N}} d\nu$$

and with $k \cdot \Delta \nu = \frac{k}{T}$

$$\eta_k = g(k \cdot \nu) = \int_{-\infty}^{\infty} f(t) \cdot e^{2\pi i \frac{kt}{T}} dt \;=\; \sum_{m=-\infty}^{\infty} \int_{(m-\frac{1}{2})T}^{(m+\frac{1}{2})T} f(t) \cdot e^{2\pi i \frac{kt}{T}} dt \;=\; \int_{\frac{-T}{2}}^{\frac{T}{2}} f_p(t) \cdot e^{2\pi i \frac{kt}{T}} dt \quad .$$

6 FOURIER SERIES OF PERIODIC FUNCTIONS

Periodic functions have (infinite) discrete Fourier series.

$$f_p(t) \;=\; \sum_{k=-\infty}^{\infty} \alpha_k \cdot e^{2\pi i \frac{kt}{T}} \quad \text{with} \quad \alpha \;=\; \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} f_p(t) e^{2\pi i \frac{kt}{T}} dt \;=\; \frac{1}{T} \cdot \eta_k \quad \text{and}$$

$$g_p(\nu) = \sum_{j=-\infty}^{\infty} \beta_j \cdot e^{2\pi i \frac{j\nu}{N}} \quad \text{with} \quad \beta_j \;=\; \frac{1}{N} \int_{\frac{-N}{2}}^{\frac{N}{2}} g_p(\nu) \cdot e^{2\pi i \frac{j\nu}{N}} d\nu \;=\; \frac{1}{N} \cdot \xi_j \quad .$$

Therefore the periodic functions can be expressed as a Fourier series using the samples of the original functions:

$$f_p(t) \;=\; \frac{1}{T} \sum_{k=-\infty}^{\infty} \eta_k \cdot e^{2\pi i \frac{kt}{T}} \quad \text{and}$$

$$g_p(\nu) \;=\; \frac{1}{N} \sum_{j=-\infty}^{\infty} \xi_j \cdot e^{2\pi i \frac{j\nu}{N}} \quad .$$

7 FINITE DISCRETE FOURIER TRANSFORMS

Combining the results above we find, that

$\{x_j\}_{j \in I}$ and $\{y_k\}_{k \in I}$ is a finite discrete Fourier transform pair.

$$x_j \;=\; f_p(j \Delta t) \;=\; \frac{1}{T} \sum_{k=-\infty}^{\infty} \eta_k \cdot e^{\frac{2\pi i}{n} jk} \;=\; \frac{1}{T} \sum_{k \in I} \left[\sum_{m=-\infty}^{\infty} g(k \Delta \nu + mN) \right] \cdot e^{\frac{2\pi i}{n}} \;=\; \frac{1}{T} \sum_{k \in I} g_p(k \Delta \nu) \cdot e^{\frac{2\pi i}{n} jk}$$

and thus

$$x_j = \sum_{k \in I} y_k \cdot \omega^{jk}$$

and, similarly,

$$y_k = \sum_{j \in I} x_j \cdot \omega^{-jk} \quad .$$

8 SUMMARY

If f and g are a continuous Fourier transform pair, then the discrete samples of the periodically continued function f_p and g_p are a finite discrete Fourier transform pair. No “aliasing” occurs (i.e. $x_j = \xi_j$ and $y_k = \eta_k$) if f and g are zero for values of t outside of $[-\frac{T}{2}, \frac{T}{2})$ and ν outside $[-\frac{N}{2}, \frac{N}{2})$.

As x_j and y_k are samples of the periodic continuation, we can express the finite discrete Fourier transform as a sum over $\mathbb{Z}_n$ instead of $I = [-\frac{n}{2}, \frac{n}{2}) \in \mathbb{Z}$:

$$y_k = \frac{1}{T} \sum_{j=0}^{n-1} x_j \omega^{-jk} \quad \text{and} \quad x_j = \frac{1}{N} \sum_{k=0}^{n-1} y_k \omega^{jk} \quad .$$

One can choose N and T symmetrically as $\sqrt{n}$ or set one of them to 1 and the other to n .

9 FAST FOURIER TRANSFORM

The fast Fourier transform algorithm constrains n to be a power of 2. The most important use one makes of the finite discrete Fourier transform is the implementation of a fast convolution (computational complexity of order $n \cdot \log(n)$). In this case, the original sequences can be extended with zeroes to a power of 2 2^l . These are transformed, pointwise multiplied and transformed back. The $2^l - n$ last elements of the result must be added to its $2^l - n$ first elements, to get the finite discrete convolution of length of an arbitrary n . Thus we have a fast algorithm for computing the convolution of two sequences of arbitrary length.

This may be used as the basis for the fast computation of a finite discrete Fourier transform. Noting that $j \cdot k = -\frac{(k-j)^2}{2} + \frac{j^2}{2} + \frac{k^2}{2}$ we find that

$$x_j = \frac{1}{N} \sum_{k=0}^{n-1} y_k \cdot \omega^{\frac{-(k-j)^2}{2}} \cdot \omega^{\frac{k^2}{2}} \cdot \omega^{\frac{j^2}{2}} = \frac{\omega^{j^2}}{N} \cdot \sum_{k=0}^{n-1} y_k \cdot \omega^{\frac{-(k-j)^2}{2}} \cdot \omega^{\frac{k^2}{2}} \quad .$$

Thus the sequence can be written as a finite convolution of the sequences $\left\{ y_k \cdot \omega^{\frac{k^2}{2}} \right\}_{k \in I}$ and $\left\{ \omega^{\frac{-k^2}{2}} \right\}_{k \in I}$.

Zürich, den 30. May 2012 Dr. sc. math. Hartwig Thomas